AREAS OF k-DIMENSIONAL NONPARAMETRIC SURFACES IN $k+1$ SPACE

BY

ROBERT N. TOMPSON

Introduction. Suppose $X \subset E_k$ ($k \geqq 2$) is a k cell, g is a real-valued continuous function on E_k, j is a positive integer not greater than k, and f is the mapping defined by the relation

$$f(x) = (x_1, \cdots, x_{j-1}, g(x), x_{j+1}, \cdots, x_k) \qquad \text{for } x \in E_k.$$

For this class of mappings on E_k into E_k (denoted by Ω_k) we concern ourselves with the validity of the formula

$$\int_X |D_j g(x)| \, d\mathcal{L}_k x = L(f|X) = \int_{E_k} S(f, X, y) d\mathcal{L}_k y = \int_{E_k} N(f, X, y) d\mathcal{L}_k y$$

where $D_j g$ is the partial derivative of g in the direction of the jth base vector, $L(f|X)$ is the Lebesgue area of the surface $f|X$, $S(f, X, y)$ and $N(f, X, y)$ are respectively the stable multiplicity and multiplicity of f on X, and $\mathcal{L}_k$ denotes the k-dimensional Lebesgue measure.

The main results, which comprise a complete theory of area for the class Ω_k, are embodied in Theorems 2.6, 2.13, 2.14, and 2.15.

For $k=2$ the results are in most part known (see [T1], [T2], and [R]).

The theory of area of the class Ω_k is intimately connected with the theory of area of k-dimensional nonparametric surfaces in E_{k+1}. In fact if $\bar{g}$ is the function defined by

$$\bar{g}(x) = (x_1, x_2, \cdots, x_k, g(x)) \qquad \text{for } x \in E_k,$$

and π is an orthogonal projection of E_{k+1} onto E_k, then

$$\bar{g} | X$$

is a k-dimensional nonparametric surface in E_{k+1}, and

$$\pi \circ \bar{g}$$

is a mapping of E_k into E_k. For some, but not all, orthogonal projections π, $\pi \circ \bar{g}$ is a member of Ω_k.

A reduction procedure is devised whereby the theory of area of the class Ω_k extends, in great part, to the mappings $\pi \circ \bar{g}$. A theory of area for k-dimensional nonparametric surfaces in E_{k+1} evolves; a theory in which complete information is obtained concerning the validity of the relation

Presented to the Society, June 20, 1954; received by the editors June 8, 1954.

$$\begin{aligned}\int_X J\bar{g}(x)d\mathcal{L}_k x &= L(\bar{g}\mid X)\\ &= \beta(k+1,\, k)^{-1}\int_{G_{k+1}}\int_{E_k} S(\overset{k}{p}_{k+1}\circ R\circ\bar{g},\, X,\, y)d\mathcal{L}_k y d\phi_{k+1}R\\ &= \beta(k+1,\, k)^{-1}\int_{G_{k+1}}\int_{E_k} N(\overset{k}{p}_{k+1}\circ R\circ\bar{g},\, X,\, y)d\mathcal{L}_k y d\phi_{k+1}R\end{aligned}$$

where $J\bar{g}$ is the Jacobian associated with $\bar{g}$ by means of its approximate differential, and where the last two members of the string are respectively the stable integralgeometric and integralgeometric areas of the surface $\bar{g}\mid X$.

The main results are contained in Theorems 3.8, 3.11, 3.15, 3.16, and 3.17.

For $k=2$ the theory of area for k-dimensional nonparametric surfaces in E_{k+1} is well established (see [F4, 6] and [S, V]).

1. Definitions.

1.1 DEFINITION. If f is a function, then inv f is its inverse, and for any set A, $f\mid A$ is the function with domain ($A\cap$domain f) for which

$$(f\mid A)(x) = f(x) \qquad \text{whenever } x\in(A\cap\text{domain } f).$$

Furthermore

$$N(f,\, A,\, x)$$

is the number (possibly ∞) of elements of the set $A\cap\{z\mid f(z)=x\}$, and

$$f^*(A) = \{x\mid x = f(z) \text{ for some } z\in A\}.$$

If g is also a function, then

$$f\circ g,$$

the superposition of f on g, is defined by the formula

$$(f\circ g)(x) = f(g(x)) \qquad \text{for all } x.$$

1.2 DEFINITION. Euclidean n space will be denoted by E_n. The usual metric and inner product (denoted by $\bullet$) are assumed for this n-dimensional vector space. We write

$$x = (x_1,\, x_2,\, \cdots,\, x_n) \qquad \text{for } x\in E_n.$$

For m and n positive integers we shall often identify $E_m\times E_n$ with E_{m+n}. Lebesgue n dimensional measure over E_n is denoted by $\mathcal{L}_n$.

$$\alpha(n) = \mathcal{L}_n(E_n\cap\{x\mid |x| < 1\}).$$

1.3 DEFINITION. If f is a function on E_n to E_1 and $x\in E_n$, then we define

$$\limsup_{z\to x}\text{ap}\, f(z)$$

to be the infimum of the numbers of the form

$$\limsup_{z\to x\ z\in A} f(z)$$

where A is a Lebesgue measurable subset of E_n with density one at x.

1.4 DEFINITION. For m and n positive integers a function L on E_m to E_n is linear if and only if

$$L(x+y) = L(x) + L(y) \qquad \text{for } x \in E_m,\ y \in E_m,$$
$$L(\lambda x) = \lambda L(x) \qquad \text{for } x \in E_m \text{ and } \lambda \text{ a real number.}$$

For $1 \leqq i \leqq m$ let ${}^mI^i$ denote the ith unit vector of E_m. Then if L is a linear function on E_m to E_n

$$L({}^mI^i) = L^i = (L_1^i, L_2^i, \cdots, L_n^i) \in E_n.$$

The matrix of L, which we identify with L, is the n by m matrix whose entry in the jth row and ith column $(1 \leqq j \leqq n,\ 1 \leqq i \leqq m)$ is L_j^i. If $m \leqq n$ the square root of the sum of the squares of the determinants of all m by m minors will be denoted by $\Delta(L)$.

If $1 \leqq m \leqq n$, f is a function on E_m to E_n, and L is a linear function on E_m to E_n for which

$$\limsup_{z\to x} \mathrm{ap} \frac{|f(z) - f(x) - L(z-x)|}{|z-x|} = 0,$$

then L is unique and is termed the *approximate differential of f at x*. If x is a point at which f has the approximate differential L we denote

$$Jf(x) = \Delta(L).$$

If f is a function on E_1 into E_n and $a < b$, then $T_{t=a}^b f(t)$ is the supremum of the numbers of the form

$$\sum_{i=1}^n |f(t_i) - f(t_{i-1})|,$$

where $a = t_0 \leqq t_1 \leqq \cdots \leqq t_n = b$.

If f is a function on E_n to E_1 and j is a positive integer no greater than n, then D_jf is the function on E_n such that

$$D_jf(x) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - f(x)}{h} \qquad \text{for } x \in E_n.$$

1.5 DEFINITION. If $m \leqq n$ are positive integers, then p_n^m is the function on E_n onto E_m defined by

$$p_n^m(x) = (x_1, \cdots, x_m) \qquad \text{for } x \in E_n.$$

1.6 Definition. If n is a positive integer, then G_n will denote the set of all linear transformations R on E_n to E_n for which

$$|R(x)| = |x| \qquad \text{whenever } x \in E_n.$$

With respect to the topology of uniform convergence and the operation of superposition, G_n is a compact topological group, in fact, the orthogonal group of E_n.

The identity element of G_n will be designated by ${}^nI$, and ϕ_n will be the unique Haar measure over G_n for which $\phi_n(G_n)=1$.

The following fact may be inferred (see [W, 8]):

If f is a ϕ_n measurable function on G_n then

$$\int_{G_n} f(R)d\phi_n R = \int_{G_n} f(\text{inv } R)d\phi_n R = \int_{G_n} f(R \circ S)d\phi_n R = \int_{G_n} f(S \circ R)d\phi_n R$$

whenever $S \in G_n$.

1.7 Definition. If $m \leqq n$ are positive integers, then

$$\beta(n, m) = \frac{\alpha(m)\cdot\alpha(n-m)}{\alpha(n)\cdot\binom{n}{m}}.$$

1.8 Definition. A function g on E_n is said to be a gauge over E_n if and only if domain $g \subset \{X \mid X \subset E_n\}$, range $g \subset \{t \mid 0 \leqq t \leqq \infty\}$.

If g is a gauge over E_n and $0 < r \leqq \infty$, the function g_r is defined by the relation

$$g_r(A) = \inf_{F \in B} \sum_{S \in F} g(S) \qquad \text{for } A \subset E_n,$$

where $F \in B$ if and only if F is a countable subfamily of domain g for which

$$A \subset \bigcup_{S \in F} S, \qquad \text{diameter } S < r, \text{ whenever } S \in F.$$

One says that ϕ is generated by g if and only if g is a gauge over E_n and ϕ is the function defined by

$$\phi(A) = \lim_{r \to 0+} g_r(A) \qquad \text{for } A \subset E_n.$$

It may be shown that ϕ is a (Carathéodory outer) measure over E_n and that closed subsets of E_n are ϕ measurable.

1.9 Definition. If $m \leqq n$ are positive integers and γ_n^m, ζ_n^m, and χ_n^m are the gauges over E_n defined by

$$\gamma_n^m(S) = \sup_{R \in G} \mathcal{L}_k[(p_n^m \circ R)^*(S)]$$

whenever S is a Borel subset of E_n,

$$\zeta_n^m(S) = \beta(n, m)^{-1} \int_{G_n} \mathcal{L}_k[(p_n^m \circ R)^*(S)]d\phi_n R$$

whenever S is an analytic subset of E_n,

$$\chi_n^m(S) = \alpha(m)2^{-m} (\text{diameter } S)^m \qquad \text{whenever } S \subset E_n,$$

then $\mathcal{H}_n^m$ generated by χ_n^m, $\mathcal{J}_n^m$ generated by ζ_n^m, and Γ_n^m generated by γ_n^m are respectively the *Hausdorff*, the *integralgeometric*, and the *Gross* k-dimensional measures over n space (see [F4], [H], [C]).

One may easily check that $\mathcal{H}_n^m$, $\mathcal{J}_n^m$, and Γ_n^m are invariant under isometries of E_n and that any subset of E_n is contained in a G_δ set of equal $\mathcal{H}_n^m$ measure, in an analytic set of equal $\mathcal{J}_n^m$ measure, and in a Borel set of equal Γ_n^m measure. The equality of $\mathcal{J}_n^n$ and $\mathcal{L}_n$ is apparent from the definition. It is also true that $\mathcal{H}_n^n = \mathcal{L}_n$ (see [SD]).

1.10 Definition. If $m \leqq n$ are positive integers and $X \subset E_m$ is an m cell or its interior, then $C_n(X)$ will denote the set of continuous functions on X to E_n.

If $g \in C_n(X)$, then g is a polyhedron if and only if X can be so triangulated that g maps each simplex baracentrically onto a rectilinear simplex of E_n.

It is to be noted that relative to the topology of uniform convergence the class of polyhedra is dense in $C_n(X)$, and also that all areas used in this paper are equivalent on the class of polyhedra.

1.11 Definition. Suppose $m \leqq n$ are positive integers and f is a continuous function on E_m to E_n.

If $X \subset E_m$ is an m cell, then

$L(f|X)$, the *m-dimensional Lebesgue area* of $f|X$,

is the lower limit of the areas of polyhedra approximating $f|X$.

If $X \subset E_m$ is the interior of an m cell, then

$L(f|X)$, the *m-dimensional Lebesgue area* of $f|X$,

is the supremum of $L(f|Y)$ for all subsets Y of X which are m cells.

If $X \subset E_m$ is either an m cell or its interior, then

the *m-dimensional Hausdorff area* of $f|X = \int_{E_n} N(f, X, y)d\mathcal{H}_n^m y$;

the *m-dimensional integralgeometric area* of $f|X = \int_{E_n} N(f, X, y)d\mathcal{J}_n^m y$

$$= \beta(n, m)^{-1} \int_{G_n} \int_{E_m} N(p_n^m \circ R \circ f, X, x)d\mathcal{L}_m x d\phi_n R;$$

the *m-dimensional integralgeometric stable area* of $f \mid X$

$$= \beta(n, m)^{-1} \int_{G_n} \int_{E_m} S(p_n^m \circ R \circ f, X, x) d\mathcal{L}_m x d\phi_n R;$$

where $S(f, X, x)$ is the stable multiplicity as defined in [F7].

1.12 DEFINITION. If n is a positive integer exceeding one, j is a positive integer no greater than n, $A \subset E_n$, $z \in E_n$, and $u \in E_{n-1}$, then

$$\begin{aligned} A_z &= E_n \cap \{x + z \mid x \in A\}, \\ A_{(j)} &= E_{n-1} \cap \{w \mid (w_1, \cdots, w_{j-1}, v, w_{j+1}, \cdots, w_n) \in A \text{ for some } v\}, \\ A^u_{(j)} &= E_1 \cap \{v \mid (u_1, \cdots, u_{j-1}, v, u_{j+1}, \cdots, u_n) \in A\}. \end{aligned}$$

If f is a continuous real-valued function on E_n and i is a positive integer, then

$$K_i = E_n \cap \{x \mid |x| \leqq i^{-1}\},$$

and f_i, the ith *integral mean*, of f, is the real-valued function on E_n defined by the formula

$$f_i(x) = \alpha(n)^{-1} i^n \int_{K_i} f(x + z) d\mathcal{L}_n z \qquad \text{for } x \in E_n.$$

2. **On a certain class of mappings of E_k into E_k ($k > 1$).** Let Ω_k denote the class of mappings on E_k into itself defined by: $f \in \Omega_k$ if and only if for some positive integer j not exceeding k and for some continuous real-valued function g on E_k, f is defined by the formula

$$f(x) = (x_1, \cdots, x_{j-1}, g(x), x_{j+1}, \cdots, x_k) \qquad \text{for } x \in E_k.$$

For such a function $f \in \Omega_k$ and i a positive integer, $\dot{f}_i$ will be defined as that element of Ω_k for which

$$\dot{f}_i(x) = (x_1, \cdots, x_{j-1}, g_i(x), x_{j+1}, \cdots, x_k) \qquad \text{for } x \in E_k.$$

2.1 SECTIONAL ASSUMPTION. In the development of a theory of area for the class Ω_k it will be convenient to fix a function $f \in \Omega_k$ which will be defined by $f(x) = (x_1, \cdots, x_{k-1}, g(x))$ for $x \in E_k$, where g is a (fixed) real-valued continuous function on E_k.

No restriction in generality will be effected by this assumption (see Remark 2.20).

2.2 LEMMA. *If $Y \subset E_k$ is a bounded open convex set and i is a positive integer, then*

$$\int_{E_k} N(\dot{f}_i, Y, x) d\mathcal{L}_k x \leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(f, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z.$$

Proof. We let

$$a(u) = \inf Y^u_{(k)}, \qquad b(u) = \sup Y^u_{(k)} \qquad\qquad \text{for } u \in Y_{(k)},$$

and for each $u \in Y_{(k)}$ we define the function

$$h_u\colon \text{closure } Y^u_{(k)} \to E_1,$$
$$h_u(v) = g(u_1, u_2, \cdots, u_{k-1}, v) \qquad \text{for } v \in \text{closure } Y^u_{(k)}.$$

The remainder of the proof is divided into four parts.

PART 1. *$N(f, Y, (u, v)) = N(h_u, Y^u_{(k)}, v)$ for $(u, v) \in E_{k-1} \times E_1$.*

Proof. If A is a subset of E_k we shall denote by $q(A)$ the number (possibly ∞) of elements of A. Whence

$$\begin{aligned} N(f, Y, (u, v)) &= q\big[(E_{k-1} \times E_1) \cap \{(w, t) \mid f(w, t) = (u, v)\}\big] \\ &= q\big[(E_{k-1} \times E_1) \cap \{(w, t) \mid w = u, t \in Y^w_{(k)}, h_w(t) = v\}\big] \\ &= q\big[E_1 \cap \{t \mid t \in Y^u_{(k)}, h_u(t) = v\}\big] \\ &= N(h_u, Y^u_{(k)}, v). \end{aligned}$$

PART 2. $\int_{E_k} N(f, Y, (u, v))\, d\mathcal{L}_k(u, v) = \int_{Y_{(k)}} T^{b(u)}_{v=a(u)} g(u, v)\, d\mathcal{L}_{k-1} u.$

Proof. Using Part 1 and [F1, 4.3] we compute:

$$\begin{aligned} \int_{E_k} N(f, Y, (u, v))\, d\mathcal{L}_k(u, v) &= \int_{E_{k-1}} \int_{E_1} N(f, Y, (u, v))\, d\mathcal{L}_1 v\, d\mathcal{L}_{k-1} u \\ &= \int_{Y_{(k)}} \int_{E_1} N(h_u, Y^u_{(k)}, v)\, d\mathcal{L}_1 v\, d\mathcal{L}_{k-1} u \\ &= \int_{Y_{(k)}} \int_{E_1} N(h_u, \text{closure } Y^u_{(k)}, v)\, d\mathcal{L}_1 v\, d\mathcal{L}_{k-1} u \\ &= \int_{Y_{(k)}} T^{b(u)}_{v=a(u)} h_u(v)\, d\mathcal{L}_{k-1} u \\ &= \int_{Y_{(k)}} T^{b(u)}_{v=a(u)} g(u, v)\, d\mathcal{L}_{k-1} u. \end{aligned}$$

PART 3. *If c and d are real numbers with*

$$c < d \quad \textit{and} \quad z = (z', z'') \in E_{k-1} \times E_1,$$

then

$$T^d_{v=c} g_i(u, v) \leqq \alpha(k)^{-1} i^k \int_{K_i} T^d_{v=c} g(u + z', v + z'')\, d\mathcal{L}_k z.$$

Proof. Let $c = v_0 < v_1 < \cdots < v_m = d$. Then

$$\sum_{p=1}^{m} | g_i(u, v_p) - g_i(u, v_{p-1}) |$$

$$= \sum_{p=1}^{m} \left| \alpha(k)^{-1} i^k \int_{K_i} \{ g(u + z', v_p + z'') - g(u + z', v_{p-1} + z'') \} d\mathcal{L}_k z \right|$$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} \sum_{p=1}^{m} | g(u + z', v_p + z'') - g(u + z', v_{p-1} + z'') | \, d\mathcal{L}_k z$$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} T_{v=c}^{d} g(u + z', v + z'') d\mathcal{L}_k z.$$

PART 4. $\int_{E_k} N(\dot{f}_i, Y, x) d\mathcal{L}_k x \leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(f, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z$.

Proof. Suppose $z = (z', z'') \in E_{k-1} \times E_1$. We know that

$$Y = (E_{k-1} \times E_1) \cap \{ (u, v) \mid v \in Y_{(k)}^{u} \}.$$

It is easy to check that

$$Y_z = (E_{k-1} \times E_1) \cap \{ (u, v) \mid v - z'' \in Y_{(k)}^{u-z'} \}, \qquad [Y_{(k)}]_{z'} = [Y_z]_{(k)}.$$

Letting $x = (u, v) \in E_{k-1} \times E_1$ and applying Part 2 to $\dot{f}_i$ we infer with the help of Parts 3 and 2 that

$$\int_{E_k} N(\dot{f}_i, Y, x) d\mathcal{L}_k x = \int_{Y(k)} T_{v=a(u)}^{b(u)} g_i(u, v) d\mathcal{L}_{k-1} u$$

$$\leqq \int_{Y(k)} \alpha(k)^{-1} i^k \int_{K_i} T_{v=a(u)}^{b(u)} g(u + z', v + z'') d\mathcal{L}_k z d\mathcal{L}_{k-1} u$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{Y(k)} T_{v=a(u)}^{b(u)} g(u + z', v + z'') d\mathcal{L}_{k-1} u d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{[Y(k)]z'} T_{v=a(u-z')}^{b(u-z')} g(u, v + z'') d\mathcal{L}_{k-1} u d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{[Y_z](k)} T_{v=a(u-z')+z''}^{b(u-z')+z''} g(u, v) d\mathcal{L}_{k-1} u d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(f, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z.$$

2.3 LEMMA. *If $X \subset E_k$ is a k cell with boundary $\dot{X}$, $F \in C_k(X)$, $G \in C_k(X)$, and if $F | \dot{X}$ is homotopic to $G | \dot{X}$ where*

$$F \mid \dot{X} \text{ is a map of } \dot{X} \text{ into } E_k - \{y\},$$

$$G \mid \dot{X} \text{ is an essential map of } \dot{X} \text{ into } E_k - \{y\},$$

then y is a stable value of $F | X - \dot{X}$.

Proof. Let $r=\inf_{x\in\dot{X}} |F(x)-y|$, and let π denote the function on $E_k-\{y\}$ into the $k-1$ sphere defined by the formula

$$\pi(x) = \frac{x-y}{|x-y|} \qquad \text{for } x \in E_k - \{y\}.$$

The proof will be divided into three parts.

PART 1. *If degree* $(\pi \circ F|\dot{X})\neq 0$, *then there exists a* $p\in X-\dot{X}$ *for which* $F(p)=y$.

Proof. Since X is a k cell, for $q\in X$ we may define a continuous contraction ϕ of X into $\{q\}$:

$$\phi: \{t \mid 0 \leqq t \leqq 1\} \to \{H \mid H \text{ is a continuous function on } X \text{ into } X\},$$
$$(\phi(0))(x) = x \qquad \text{for } x \in X,$$
$$(\phi(1))(x) = q \qquad \text{for } x \in X.$$

Suppose that the statement is false. Then for $0\leqq t\leqq 1$

$$F \circ \phi(t) \mid \dot{X}: \dot{X} \to E_k - \{y\},$$
$$F \circ \phi(t) \mid \dot{X} \text{ is homotopic to } F \circ \phi(0) \mid \dot{X},$$
$$\pi \circ F \circ \phi(t) \mid \dot{X} \text{ is homotopic to } \pi \circ F \circ \phi(0) \mid \dot{X}.$$

Accordingly we arrive at the contradiction

$$0 = \text{degree}\,(\pi \circ F \circ \phi(1) \mid \dot{X}) = \text{degree}\,(\pi \circ F \circ \phi(0) \mid \dot{X}) = \text{degree}\,(\pi \circ F \mid \dot{X}) \neq 0.$$

PART 2. *If* $H\in C_k(X)$ *and* $|H|\dot{X}-F|\dot{X}|<r$, *then*

$$\text{degree}\,(\pi \circ H \mid \dot{X}) = \text{degree}\,(\pi \circ F \mid \dot{X}).$$

Proof. Since for $x\in\dot{X}$

$$|F(x) - H(x)| < |F(x) - y|,$$

it follows that

$$y \notin E_k \cap \{z \mid z = F(x) + t(H(x) - F(x)) \text{ for } 0 \leqq t \leqq 1\}.$$

Hence it suffices to define the continuous function

$$\phi: \dot{X} \times \{t \mid 0 \leqq t \leqq 1\} \to E_k - \{y\},$$
$$\phi(x, t) = F(x) + t(H(x) - F(x)) \qquad \text{for } (x, t) \in \dot{X} \times \{t \mid 0 \leqq t \leqq 1\},$$

to establish that $F|\dot{X}$ and $H|\dot{X}$ are homotopic. Consequently $\pi \circ F|\dot{X}$ and $\pi \circ H|\dot{X}$ are homotopic and

$$\text{degree}\,(\pi \circ F \mid \dot{X}) = \text{degree}\,(\pi \circ H \mid \dot{X}).$$

PART 3. *y is a stable value of F.*

Proof. We infer from the hypothesis that

$$\text{degree } (\pi \circ F \mid \dot{X}) = \text{degree } (\pi \circ G \mid \dot{X}) \neq 0.$$

If $L \in C_k(X)$ and $|L-F| < r$, we may use Part 2 to establish that

$$\text{degree } (\pi \circ L \mid \dot{X}) = \text{degree } (\pi \circ F \mid \dot{X}),$$

and Part 1 to show the existence of a $p \in X - \dot{X}$ for which $L(p) = y$.

2.4 THEOREM. *If Y is a bounded open convex subset of E_k, then*

$$N(f, Y, x) = S(f, Y, x) \quad \text{for } \mathcal{L}_k \text{ almost all } x \in E_k.$$

Proof. We assume h_u to have the same meaning as in Lemma 2.2. Let C be the set of points (u, v) in Y for which h_u has either a relative maximum or a relative minimum at v. We may check that C is a F_σ set and

$$\overset{*}{h}_u(\overset{u}{C}_{(k)}) \text{ is countable for } u \in Y_{(k)}.$$

If follows that $f^*(C)$ is $\mathcal{L}_k$ measurable and

$$\mathcal{L}_k[f^*(C)] = 0.$$

We know that

$$N(f, Y - C, x) = N(f, Y, x) \qquad \text{for } x \in f^*(Y - C) - f^*(C),$$
$$N(f, Y, x) = S(f, Y, x) \qquad \text{for } x \notin f^*(Y),$$

hence we may complete the proof by showing that f is stable [F4, 6.6] at every point of $Y - C$. For if this were so, then

$$S(f, Y, x) \leqq N(f, Y, x) = N(f, Y - C, x) \leqq S(f, Y, x)$$

whenever $x \in f^*(Y-C) - f^*(C)$.

Let $\epsilon > 0$.

If $(u^0, v^0) = (u_1^0, u_2^0, \cdots, u_{k-1}^0, v^0) \in Y - C$, then using the continuity of g we can select

$$u^1 \in Y_{(k)}, \qquad u^2 \in Y_{(k)}, \qquad v^1 \in E_1, \qquad v^2 \in E_1,$$

satisfying

$$\overset{0}{u_i} - \epsilon < \overset{1}{u_i} < \overset{0}{u_i} < \overset{2}{u_i} < \overset{0}{u_i} + \epsilon \qquad \text{for } i = 1, 2, \cdots, k-1,$$
$$\overset{0}{v} - \epsilon < \overset{1}{v} < \overset{0}{v} < \overset{2}{v} < \overset{0}{v} + \epsilon,$$

and such that if we denote

$$P = E_{k-1} \cap \{u \mid \overset{1}{u_i} \leqq u_i \leqq \overset{2}{u_i} \text{ for } i = 1, 2, \cdots, k-1\},$$
$$X = (E_{k-1} \times E_1) \cap \{(u, v) \mid u \in P \text{ and } v^1 \leqq v \leqq v^2\},$$

then $X \subset Y$ and either

(1) $$h_u(v^1) < h_u(v^0) < h_u(v^2) \qquad \text{whenever } u \in P,$$

or else

$$(2) \qquad h_u(v^1) > h_u(v^0) > h_u(v^2) \qquad \text{whenever } u \in P.$$

Observe that X is a k cell with diameter less than $2k^{1/2}\epsilon$ and, denoting its boundary by $\dot{X}$ and $f(u^0, v^0)$ by y, that $f|\dot{X}$ is a map of $\dot{X}$ into $E_k - \{y\}$.

If (1) occurs we can define the function G on $\dot{X}$ into $E_k - \{y\}$ by

$$G(u, v) = (u, y_k + v - v^0) \qquad \text{for } (u, v) \in (E_{k-1} \times E_1) \cap \dot{X},$$

and the function ϕ on $\dot{X} \times \{t | 0 \leqq t \leqq 1\}$ into $E_k - \{y\}$ by

$$\phi(u, v, t) = (u, (1 - t)(y_k + v - v^0) + th_u(v))$$

for $(u, v, t) \in ((E_k \times E_1) \cap \dot{X}) \times \{t | 0 \leqq t \leqq 1\}$, and infer that

G is an essential map of $\dot{X}$ into $E_k - \{y\}$,

G is homotopic to $f | \dot{X}$.

Lemma 2.3 implies that y is a stable value of $f | X - \dot{X}$, and from the arbitrary nature of ϵ we conclude that f is stable at (u^0, v^0).

Whenever (2) occurs a similar treatment is employed. The proof is complete.

2.5 Lemma. *If $X \subset E_k$ is a k cell with boundary $\dot{X}$, then* $\mathcal{L}_k[f^*(\dot{X})] = 0$.

Proof. Let

$$h(x) = (x_1, x_2, \cdots, x_k, g(x)) \qquad \text{for } x \in X.$$

Then observing that $\dot{X}$ lies on $2k$ $k-1$ cells A_i $(i = 1, 2, \cdots, 2k)$ it is apparent that the h image of each such $k-1$ cell lies on a k plane in E_{k+1}, and that

$$\mathcal{H}_{k+1}^{k}[h^*(\dot{X})] \leqq \sum_{i=1}^{2k} \mathcal{H}_{k+1}^{k}[h^*(A_i)].$$

If j is a positive integer no greater than $2k$, let π be an isometric projection onto E_k of the k plane containing A_j and $h^*(A_j)$, which satisfies

$$\pi^*(A_j) \subset E_k \cap \{x \mid x_k = 0\}.$$

It follows that

$$\text{the number of elements in } [(\pi \circ h)^*(A_j)]_{(k)}^{u} = 1$$

for $u \in [(\pi \circ h)^*(A_j)]_{(k)}$. We apply Fubini's theorem to obtain

$$\begin{aligned} \mathcal{H}_{k+1}^{k}[h^*(A_j)] &= \mathcal{L}_k[(\pi \circ h)^*(A_j)] \\ &= \int_{[(\pi \circ h)^*(A_j)]_{(k)}} \mathcal{L}_1\{[(\pi \circ h)^*(A_j)]_{(k)}^{u}\} d\mathcal{L}_{k-1}u = 0, \\ \mathcal{H}_{k+1}^{k}[h^*(\dot{X})] &= 0. \end{aligned}$$

Finally letting R be the element of G_k for which

$$R(w) = (w_1, \cdots, w_{k-1}, w_{k+1}, w_k) \qquad \text{for } w \in E_{k+1},$$

we conclude from the Lipschitzian character of $p_{k+1}^k \circ R$ that

$$\mathcal{L}_k[f^*(\dot{X})] = \mathcal{L}_k[(p_{k+1}^k \circ R \circ h)^*(\dot{X})] = 0.$$

2.6 THEOREM. *If $X \subset E_k$ is a k cell, then*

$$\int_{E_k} N(f, X, x) d\mathcal{L}_k x = \int_{E_k} S(f, X, x) d\mathcal{L}_k x = L(f \mid X) = \lim_{j \to \infty} L(\dot{f}_j \mid X).$$

Proof. Let Y denote the interior of X and let A be an open interval of E_k for which

$$\text{closure } A \subset Y.$$

Select a positive integer i so large that

$$A_z \subset Y \qquad \text{for } z \in K_i.$$

Then using [F4, 6.13], [F2, 4.5] and Lemma 2.2 we compute

$$\begin{aligned} L(\dot{f}_i \mid \text{closure } A) &= \int_A J\dot{f}_i(x) d\mathcal{L}_k x \\ &= \int_{E_k} N(\dot{f}_i, A, x) d\mathcal{L}_k x \\ &\leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(f, A_z, x) d\mathcal{L}_k x d\mathcal{L}_k z \\ &\leqq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x. \end{aligned}$$

It follows using the lower semi-continuity of L on $C_k(X)$ that

$$L(f \mid \text{closure } A) \leqq \liminf_{i \to \infty} L(\dot{f}_i \mid \text{closure } A) \leqq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x.$$

From the arbitrary nature of A and Theorem 2.4 we may conclude

$$L(f \mid Y) \leqq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x = \int_{E_k} S(f, Y, x) d\mathcal{L}_k x \leqq L(f \mid Y).$$

In view of Lemma 2.5 it is seen that this relation holds with Y replaced by X.

For the other part of the statement we recall in general that

$$L(f \mid X) \leqq \liminf_{j \to \infty} L(\dot{f}_j \mid X).$$

Therefore under the assumption that $L(f|X)$ is finite we need to prove that

$$\limsup_{j\to\infty} L(f_j | X) \leqq L(f | X).$$

Pick $\epsilon>0$. Utilizing the foregoing results we can select an open interval U of E_k such that

$$X \subset U, \qquad L(f | U) \leqq L(f | X) + \epsilon.$$

Let j be so large a positive integer that

$$(\text{interior } X)_z \subset U \qquad \text{whenever } z \in K_j.$$

Then we can show just as before that

$$L(f_j | X) \leqq \int_{E_k} S(f, U, x) d\mathcal{L}_k x \leqq L(f | U) \leqq L(f | X) + \epsilon.$$

Accordingly

$$\limsup_{j\to\infty} L(f_j | X) \leqq L(f | X) + \epsilon.$$

Since ϵ was arbitrary this completes the proof.

2.7 Remark. Using the results of the preceding theorem and the notation of [F 6] we find for $X \subset E_k$ a k cell that

$$\begin{aligned} L(f | X) &= M^{**}(f | X) = M^{*}(f | X) = S^{**}(f | X) = S^{*}(f | X) \\ &= V^{**}(f | X) = V^{*}(f | X) = U^{**}(f | X) = U^{*}(f | X) \\ &= N^{**}(f | X) = N^{*}(f | X). \end{aligned}$$

2.8 Definition. If $X \subset E_k$ is a k cell and $h \in C_1(X)$, then

(i) for i a positive integer no greater than k, h is said to be of *bounded variation* (i) *in the sense of Tonelli* (BVT (i)) on X if and only if

$$\int_{X_{(i)}} T_{v=\inf X_{(i)}^u}^{\sup X_{(i)}^u} h(u_1, \cdots, u_{i-1}, v, u_{i+1}, \cdots, u_k) d\mathcal{L}_{k-1} u < \infty;$$

(ii) h is said to be of *bounded variation in the sense of Tonelli* (BVT) on X if and only if h is BVT (i) on X whenever $i = 1, 2, \cdots, k$.

2.9 Definition. If $X \subset E_k$ is a k cell and $h \in C_1(X)$, then

(i) for i a positive integer no greater than k, h is said to be *absolutely continuous* (i) *in the sense of Tonelli* (ACT (i)) on X if and only if h is BVT (i) on X and the function

$$v \to h(u_1, \cdots, u_{i-1}, v, u_{i+1}, \cdots, u_k)$$

is absolutely continuous in the classical sense on $X_{(i)}^u$, for $\mathcal{L}_{k-1}$ almost all $u \in X_{(i)}$;

(ii) h is said to be *absolutely continuous in the sense of Tonelli* (ACT) on X if and only if h is ACT (i) on X whenever $i=1, 2, \cdots, k$.

2.10 DEFINITION. If $X \subset E_k$ is a k cell and $h \in C_k(X)$, then h is said to be *absolutely continuous* on X if and only if

$$\int_{E_k} N(h, X, y) d\mathcal{L}_k y < \infty,$$

and h transforms subsets of X of $\mathcal{L}_k$ measure zero into sets of $\mathcal{L}_k$ measure zero.

2.11 SECTIONAL ASSUMPTION. For the rest of this section $X \subset E_k$ will denote a k cell.

2.12 LEMMA. *If g is $BVT(k)$ on X, then*

$$\begin{aligned}\int_Y \left| D_k g(x) \right| d\mathcal{L}_k x &= \int_{Y(k)} \int_{Y(k)^u} \left| D_k g(u, v) \right| d\mathcal{L}_1 v d\mathcal{L}_{k-1} u \\ &\leqq \int_{Y(k)} T_{v=\inf Y(k)^u}^{\sup Y(k)^u} g(u, v) d\mathcal{L}_{k-1} u \\ &= \int_{E_k} N(f, Y, (u, v)) d\mathcal{L}_k(u, v) \\ &= \int_{E_k} N(f, Y, y) d\mathcal{L}_k y\end{aligned}$$

whenever $Y \subset X$ and Y is a k cell.

Proof. This statement follows directly from Definition 2.8 and Part 2 of Lemma 2.2.

2.13 THEOREM. *The following statements are equivalent:*
(i) *g is $BVT(k)$ on X,*
(ii) $L(f|X) < \infty$.

Proof. By virtue of Part 2 of Lemma 2.2 and Theorem 2.6

$$\int_{X(k)} T_{v=\inf X(k)^u}^{\sup X(k)^u} g(u, v) d\mathcal{L}_{k-1} u = L(f \mid X).$$

The theorem is an immediate consequence of this equality.

2.14 THEOREM. *If g is $BVT(k)$ on X, U is the set of those points x in X for which $D_k g(x)$ exists, Y is an $\mathcal{L}_k$ measurable subset of X and Z is an analytic subset of E_k contained in X, then*
(i) *U is a Borel subset of E_k and $\mathcal{L}_k(X-U)=0$,*
(ii) $\int_Y |D_k g(x)| d\mathcal{L}_k x = \int_{E_k} N(f, Y \cap U, y) d\mathcal{L}_k y < \infty$,
(iii) $\int_Z |D_k g(x)| d\mathcal{L}_k x \leqq \int_{E_k} N(f, Z, y)\, d\mathcal{L}_k y < \infty$.

Proof. We know from [S, V4.1] that $D_k g$ is a Borel measurable function. We may infer that

$$U \text{ is a Borel subset of } E_k,$$

and since

$$\mathcal{L}_1[(X - U)^u_{(k)}] = 0 \qquad \text{for } \mathcal{L}_{k-1} \text{ almost all } u \in X_{(k)},$$

may use Fubini's theorem to conclude that

$$\mathcal{L}_k(X - U) = 0.$$

Using the notation of Lemma 2.2 the statement

$$| D_k g(u, v) | = | D_1 h_u(v) | = J h_u(v) \qquad \text{for } v \in U^u_{(k)}$$

is true for $u \in X_{(k)}$. Also we know that

$$(Y \cap U)^u_{(k)}$$

is an $\mathcal{L}_1$ measurable subset of E_1 for $u \in X_{(k)}$.

Upon combining these facts and using [F2, 5.2] (applied to h_u), Part 1 of Lemma 2.2, and Theorem 2.13, we find that

$$\begin{aligned}
\int_Y | D_k g(x) | \, d\mathcal{L}_k x &= \int_{Y \cap U} | D_k g(x) | \, d\mathcal{L}_k x \\
&= \int_{(Y\cap U)_{(k)}} \int_{(Y \cap U)_{(k)}} | D_k g(u, v) | \, d\mathcal{L}_1 v \, d\mathcal{L}_{k-1} u \\
&= \int_{(Y\cap U)_{(k)}} \int_{(Y \cap U)_{(k)}^u} J h_u(v) \, d\mathcal{L}_1 v \, d\mathcal{L}_{k-1} u \\
&= \int_{(Y\cap U)_{(k)}} \int_{E_1} N(h_u, (Y \cap U)^u_{(k)}, v) \, d\mathcal{L}_1 v \, d\mathcal{L}_{k-1} u \\
&= \int_{E_{k-1}} \int_{E_1} N(f, Y \cap U, (u, v)) \, d\mathcal{L}_1 v \, d\mathcal{L}_{k-1} u \\
&= \int_{E_k} N(f, Y \cap U, y) \, d\mathcal{L}_k y < \infty.
\end{aligned}$$

For (iii) it is only necessary in addition to observe that

$$N(f, Z, y) \text{ is } \mathcal{L}_k \text{ measurable in } y,$$

$$\int_{E_k} N(f, Z, y) \, d\mathcal{L}_k y < \infty.$$

2.15 THEOREM. *If g is $BVT(k)$ on X, then the following statements are equivalent:*

(i) *f is absolutely continuous on X,*

(ii) $\int_Y |D_k g(x)| d\mathcal{L}_k x = \int_{E_k} N(f, Y, y) d\mathcal{L}_k y < \infty$ *whenever Y is an $\mathcal{L}_k$ measurable subset of X,*

(iii) $\int_X |D_k g(x)| d\mathcal{L}_k x = \int_{E_k} N(f, X, y) d\mathcal{L}_k y < \infty$,

(iv) *g is $ACT(k)$ on X.*

Proof. The proof is divided into five parts.

PART 1. (i) *implies* (ii).

Proof. Let U be the set of points x in X for which $D_k g(x)$ exists, and Y be an $\mathcal{L}_k$ measurable subset of X. Then

$$N(f, Y, y) \text{ is } \mathcal{L}_k \text{ measurable in } y,$$
$$\mathcal{L}_k(Y - U) = 0,$$

and we may infer, using the preceding theorem, that

$$\begin{aligned}\int_Y |D_k g(x)| d\mathcal{L}_k x &= \int_{E_k} N(f, Y \cap U, y) d\mathcal{L}_k y \\ &= \int_{E_k} N(f, Y, y) d\mathcal{L}_k y < \infty.\end{aligned}$$

PART 2. (ii) *implies* (iii).

PART 3. (iii) *implies* (iv).

Proof. Lemma 2.12 implies that

$$T_{v=\inf X_{(k)}{}^u}^{\sup X_{(k)}{}^u} g(u, v) = \int_{X_{(k)}{}^u} |D_k g(u, v)| d\mathcal{L}_1 v$$

for $\mathcal{L}_{k-1}$ almost all $u \in X_{(k)}$. Whence

$$g \text{ is } ACT(k) \text{ on } X.$$

PART 4. (iv) *implies* (ii).

Proof. If $Y \subset X$ and Y is a k cell, then (iv) being true means

$$T_{v=\inf Y_{(k)}{}^u}^{\sup Y_{(k)}{}^u} g(u, v) = \int_{Y_{(k)}{}^u} |D_k g(u, v)| d\mathcal{L}_1 v$$

for $\mathcal{L}_{k-1}$ almost all $u \in Y_{(k)}$,

$$\int_{E_k} N(f, Y, y) d\mathcal{L}_k y < \infty.$$

We infer from Lemma 2.12 that

$$\int_Y |D_k g(x)| d\mathcal{L}_k x = \int_{E_k} N(f, Y, y) d\mathcal{L}_k y < \infty.$$

This equality may be extended to hold whenever Y is a Borel subset of E_k contained in X.

Now suppose W is an $\mathcal{L}_k$ measurable subset of X. Borel subsets S and T of E_k may be chosen in such a way that

$$S \subset W \subset T \subset X, \qquad \mathcal{L}_k(T - S) = 0.$$

Since

$$N(f, S, y) \leqq N(f, W, y) \leqq N(f, T, y) \qquad \text{for } y \in E_k,$$

$$\int_{E_k} N(f, S, y) d\mathcal{L}_k y = \int_S | D_k g(x) | d\mathcal{L}_k x = \int_W | D_k g(x) | d\mathcal{L}_k x$$

$$= \int_T | D_k g(x) | d\mathcal{L}_k x = \int_{E_k} N(f, T, y) d\mathcal{L}_k y < \infty,$$

it follows that

$$N(f, S, y) = N(f, T, y) \qquad \text{for } \mathcal{L}_k \text{ almost all } y \text{ in } E_k;$$

$N(f, W, y)$ is $\mathcal{L}_k$ measurable in y,

$$\int_W | D_k g(x) | d\mathcal{L}_k x = \int_{E_k} N(f, W, y) d\mathcal{L}_k y < \infty.$$

PART 5. (ii) *implies* (i).

2.16 REMARK. A combination of Theorems 2.6, 2.14, and 2.15 reveals that

g is $BVT(k)$ on X implies

$$\int_X | D_k g(x) | d\mathcal{L}_k x \leqq L(f | X) = \int_{E_k} S(f, X, y) d\mathcal{L}_k y = \int_{E_k} N(f, X, y) d\mathcal{L}_k y,$$

equality holding if and only if g is $ACT(k)$.

2.17 THEOREM. *If h is an $\mathcal{L}_k$ measurable function satisfying*

$$-\infty \leqq h(x) \leqq \infty$$

for $\mathcal{L}_k$ almost all x in E_k, then

g is $ACT(k)$ on X implies

$$\int_Y (h \circ f)(x) \cdot | D_k g(x) | d\mathcal{L}_k x = \int_{E_k} h(y) \cdot N(f, Y, y) d\mathcal{L}_k y$$

whenever Y is an $\mathcal{L}_k$ measurable subset of X.

Proof. This statement is an immediate consequence of Theorem 2.15 and [F3, 2.1].

2.18 THEOREM. *If for $x \in E_k$ and $r > 0$*

$$C(x, r) = E_k \cap \{z \mid |z - x| \leqq r\},$$

then

g is BVT(k) on X implies

$$\lim_{r\to 0+} \frac{L[f \mid C(x, r)]}{\alpha(k)r^k} = |D_k g(x)| \qquad \text{for } \mathcal{L}_k \text{ almost all } x \in \text{interior } X.$$

Proof. If U is the set of those points x in X for which $D_k g(x)$ exists, we know that U is a Borel subset of E_k and may define functions β, γ, and δ as follows:

$$\beta(Y) = \int_{E_k} N(f, Y, y) d\mathcal{L}_k y,$$

$$\gamma(Y) = \int_Y |D_k g(x)| d\mathcal{L}_k x,$$

$$\delta(Y) = \int_{E_k} N(f, Y - U, y) d\mathcal{L}_k y,$$

for Y a Borel subset of E_k contained in X.

Employing the terminology of Saks [S, p. 30], β, γ, and δ are additive set functions on the class of Borel subsets of E_k contained in X,

$$\gamma \text{ is } \mathcal{L}_k \text{ absolutely continuous,}$$

and since $\mathcal{L}_k(X - U) = 0$

$$\delta \text{ is } \mathcal{L}_k \text{ singular.}$$

Because of Theorem 2.14

$$\beta = \gamma + \delta.$$

Hence using Theorem 2.6 and [S, IV 5.4, 6.3, 7.1] we conclude that

$$\begin{aligned} \lim_{r\to 0+} \frac{L[f \mid C(x, r)]}{\alpha(k)r^k} &= \lim_{r\to 0+} \frac{\beta[C(x, r)]}{\alpha(k)r^k} \\ &= \lim_{r\to 0+} \frac{\gamma[C(x, r)]}{\alpha(k)r^k} + \lim_{r\to 0+} \frac{\delta[C(x, r)]}{\alpha(k)r^k} \\ &= |D_k g(x)| \end{aligned}$$

for $\mathcal{L}_k$ almost all $x \in$ interior X.

2.19 Remark. If g is BVT, then a computation shows that

$$Jf(x) = |D_k g(x)| \qquad \text{for } \mathcal{L}_k \text{ almost all } x \text{ in } X.$$

In this case the validity of Theorems 2.14, 2.15, 2.17, and 2.18 and Remark

2.16 holds whenever $|D_k g|$ is replaced by Jf.

2.20 REMARK. If $f \in \Omega_k$ were of the form

$$f(x) = (x_1, \cdots, x_{j-1}, g(x), x_{j+1}, \cdots, x_k) \qquad \text{for } x \in E_k,$$

for j some positive integer less than k and g a continuous real-valued function on E_k, then the foregoing theory would hold, provided only that the conditions BVT(k) and ACT(k) be replaced by BVT(j) and ACT(j), and the function $D_k g$ be replaced by $D_j g$.

3. **k dimensional nonparametric surfaces in E_{k+1} ($k>1$).** Each orthogonal projection of E_{k+1} onto E_k is of the form

$$\overset{k}{p}_{k+1} \circ R$$

for some $R \in G_{k+1}$. In fact $p^k_{k+1} \circ R$ maps the k-dimensional subspace of E_{k+1} spanned by the k tuple of vectors

$$\langle (\text{inv } R)^1, \cdots, (\text{inv } R)^k \rangle$$

onto E_k, transforming $(\text{inv } R)^i$ into ${}^kI^i$ for $i = 1, 2, \cdots, k$.

Since for $x \in E_{k+1}$ and $R \in G_{k+1}$,

$$(\overset{k}{p}_{k+1} \circ R)(x) = (x \bullet \overset{1}{(\text{inv } R)}, \cdots, x \bullet \overset{k}{(\text{inv } R)}),$$

two orthogonal projections $p^k_{k+1} \circ S$ and $p^k_{k+1} \circ T$ are equal if and only if

$$(\text{inv } S)^i = (\text{inv } T)^i \qquad \text{for } i = 1, 2, \cdots, k.$$

For m a positive integer greater than two, Λ_m will denote the set of all those elements R of G_m for which there exists some $S \in G_2$ satisfying

$$\begin{aligned}
\overset{i}{R} &= {}^m\overset{i}{I} && \text{for } i = 1, 2, \cdots, m-2;\\
\overset{i}{R}_j &= 0 && \text{for } i = m-1, m; j = 1, 2, \cdots, m-2;\\
\overset{i}{R}_j &= S^{i-m+2}_{j-m+2} && \text{for } i = m-1, m; j = m-1, m.
\end{aligned}$$

If f is a continuous real-valued function on E_k and i is a positive integer, then $\bar{f}$ and $\bar{f}_i$ will be the functions defined by the relations

$$\begin{aligned}
\bar{f}(x) &= (x_1, x_2, \cdots, x_k, f(x)) && \text{for } x \in E_k,\\
\bar{f}_i(x) &= (x_1, x_2, \cdots, x_k, f_i(x)) && \text{for } x \in E_k.
\end{aligned}$$

3.1 SECTIONAL ASSUMPTIONS. We let f be a fixed continuous real-valued function on E_k, and for j a positive integer no greater than $k+1$, ${}^jR$ will denote that element of G_{k+1} for which

$${}^jR(x) = (x_1, \cdots, x_{j-1}, x_{k+1}, x_{j+1}, \cdots, x_k, x_j) \qquad \text{for } x \in E_k.$$

Observe that

$$^{k+1}R \text{ is the identity element of } G_{k+1},$$

$$\overset{k}{p}_{k+1} \circ {}^{j}R \circ \bar{f} \in \Omega_k \qquad \text{for } j = 1, 2, \cdots, k+1.$$

3.2 Lemma. *If Y is a bounded open convex subset of E_k and i is a positive integer, then*

$$S \in \Lambda_{k+1} \text{ implies}$$

(i) $\int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \bar{f}_i,\ Y,\ x) d\mathcal{L}_k x$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \bar{f},\ Y_z,\ x)\ d\mathcal{L}_k x d\mathcal{L}_k z,$$

(ii) $N(\overset{k}{p}_{k+1} \circ S \circ \bar{f},\ Y, x) = S(\overset{k}{p}_{k+1} \circ S \circ \bar{f},\ Y,\ x)$ *for $\mathcal{L}_k$ almost all x in E_k.*

Proof. We know that

$$\begin{aligned} (\overset{k}{p}_{k+1} \circ S)(w) &= (w \bullet (\text{inv } S)^{1}, \cdots, w \bullet (\text{inv } S)^{k}) \\ &= (w_1, w_2, \cdots, w_{k-1}, w_k \cdot \overset{k}{S_k} + w_{k+1} \cdot \overset{k+1}{S_k}) \end{aligned}$$

for $w \in E_{k+1}$. Thus

$$\begin{aligned} (\overset{k}{p}_{k+1} \circ S \circ \bar{f})(x) &= (\overset{k}{p}_{k+1} \circ S)(x_1, x_2, \cdots, x_k, f(x)) \\ &= (x_1, x_2, \cdots, x_{k-1}, \overset{k}{S_k} \cdot x_k + \overset{k+1}{S_k} \cdot f(x)) \end{aligned}$$

whenever $x \in E_k$.

We may infer then that $\overset{k}{p}_{k+1} \circ S \circ \bar{f} \in \Omega_k$. Moreover upon letting $F(x) = \overset{k}{S_k} \cdot x_k + \overset{k+1}{S_k} \cdot f(x)$ for $x \in E_k$, a check reveals that

$$F_i(x) = \overset{k}{S_k} \cdot x_k + \overset{k+1}{S_k} \cdot f_i(x) \qquad \text{for } x \in E_k,$$

$$\cdot (\overset{k}{p}_{k+1} \circ S \circ \bar{f})_i = \overset{k}{p}_{k+1} \circ S \circ \bar{f}_i,$$

for every positive integer i. Accordingly statements (i) and (ii) follow from Lemma 2.2 and Theorem 2.4.

3.3 Lemma. *If $R \in G_{k+1}$, then there exist functions S, T and U for which*

$$S \in \Lambda_{k+1}, \qquad T \in G_{k+1}, \qquad U \in G_k,$$

$$T({}^{k+1}I^{k+1}) = {}^{k+1}I^{k+1},$$

$$\overset{k}{p}_{k+1} \circ R = U \circ \overset{k}{p}_{k+1} \circ S \circ T.$$

Proof. We know for any two vector subspaces of a vector space that the sum of their dimensions is equal to the sum of the dimensions of the vector spaces generated respectively by their union and intersection.

Accordingly if Z is the subspace of E_{k+1} spanned by the k-tuple of vectors

$$\langle {}^{k+1}I^1, {}^{k+1}I^2, \cdots, {}^{k+1}I^k \rangle$$

and K the subspace of E_{k+1} spanned by the k-tuple of vectors

$$\langle (\text{inv } R)^1, (\text{inv } R)^2, \cdots, (\text{inv } R)^k \rangle,$$

then the dimension of the subspace $K \cap Z$ equals either k or $k-1$.

If dimension $(K \cap Z) = k$, then K coincides with Z, and we may select $T \in G_{k+1}$ so that T is the rotation of Z which transforms $(\text{inv } R)^i$ into ${}^{k+1}I^i$ for $i = 1, 2, \cdots, k$. Letting U denote the identity map of E_k and S the identity map of E_{k+1} we may easily check that

$$\overset{k}{p}_{k+1} \circ R = U \circ (\overset{k}{p}_{k+1} \circ R \circ \text{inv } T) \circ T,$$
$$\overset{k}{p}_{k+1} \circ R \circ \text{inv } T = \overset{k}{p}_{k+1} \circ S.$$

Now suppose that the dimension of $K \cap Z$ is $k-1$.

Using the transitivity of the orthogonal group, we can choose functions A, U, and T for which

$$A \in G_{k+1}, \qquad A^*(K) = K,$$
$$\langle (A \circ \text{inv } R)^1, (A \circ \text{inv } R)^2, \cdots, (A \circ \text{inv } R)^{k-1} \rangle \text{ spans } K \cap Z,$$
$$U \in G_k,$$
$$U[(\overset{k}{p}_{k+1} \circ R)((\text{inv } A \circ \text{inv } R)^i)] = {}^kI^i \qquad \text{for } i = 1, 2, \cdots, k,$$
$$T \in G_{k+1}, \qquad T^*(Z) = Z,$$
$$T[(A \circ \text{inv } R)^i] = {}^{k+1}I^i \quad \text{for } i = 1, 2, \cdots, k-1.$$

Since $(T \circ A \circ \text{inv } R)^k$ lies in the orthogonal complement of the space generated by $\langle {}^{k+1}I^1, {}^{k+1}I^2, \cdots, {}^{k+1}I^{k-1} \rangle$, an element B of G_2 may be picked such that

$$(T \circ A \circ \text{inv } R)^k_{k-1+i} = B^1_i \qquad \text{for } i = 1, 2.$$

Then defining S to be the element of G_{k+1} satisfying

$$(\text{inv } S)^i = {}^{k+1}I^i \qquad \text{for } i = 1, 2, \cdots, k-1;$$
$$(\text{inv } S)^i_j = 0 \qquad \text{for } i = k, k+1; j = 1, 2, \cdots, k-1;$$
$$(\text{inv } S)^i_j = B^{i-k+1}_{j-k+1} \qquad \text{for } i = k, k+1; j = k, k+1;$$

we may infer that S (along with inv S) is an element of Λ_{k+1}.

Inasmuch as

$$\begin{aligned}(\text{inv } (S \circ T))^i &= (\text{inv } T \circ \text{inv } S)^i \\ &= \text{inv } T[(\text{inv } S)^i] \\ &= \text{inv } T[(T \circ A \circ \text{inv } R)^i] \\ &= (A \circ \text{inv } R)^i \\ &= (\text{inv } (R \circ \text{inv } A))^i \qquad \text{for } i = 1, 2, \cdots, k,\end{aligned}$$

it follows that

$$\overset{k}{p}_{k+1} \circ S \circ T = \overset{k}{p}_{k+1} \circ R \circ \operatorname{inv} A.$$

Let i be a positive integer no greater than k. To complete the proof it is sufficient to show the equality of the functions $\overset{k}{p}_{k+1} \circ R$ and $U \circ \overset{k}{p}_{k+1} \circ S \circ T$ on the vector

$$(\operatorname{inv} R)^i.$$

We compute:

$$\begin{aligned}(U \circ \overset{k}{p}_{k+1} \circ S \circ T)[(\operatorname{inv} R)^i] &= (U \circ \overset{k}{p}_{k+1} \circ R \circ \operatorname{inv} A)[(\operatorname{inv} R)^i] \\ &= U[(\overset{k}{p}_{k+1} \circ R)((\operatorname{inv} A \circ \operatorname{inv} R)^i)] \\ &= {}^k I^i.\end{aligned}$$

3.4 LEMMA. *If Y is a bounded open convex subset of E_k and i is a positive integer, then*

$$R \in G_{k+1} \textit{ implies}$$

$$\int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}_i, Y, x) d\mathcal{L}_k x$$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \bar{f}, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z.$$

Proof. Suppose $R \in G_{k+1}$. We use the preceding lemma to select functions U, S, T, t such that

$$U \in G_k, \qquad S \in \Lambda_{k+1}, \qquad T \in G_{k+1}, \qquad t \in G_k,$$
$$T({}^{k+1}I^{k+1}) = {}^{k+1}I^{k+1},$$

if $w \in E_k$, $z \in E_k$, and $v \in E_1$, then $t(z) = w$ if and only if $T(z_1, z_2, \cdots, z_k, v) = (w_1, w_2, \cdots, w_k, v)$,

$$\overset{k}{p}_{k+1} \circ R = U \circ \overset{k}{p}_{k+1} \circ S \circ \operatorname{inv} T.$$

We shall denote

$$\operatorname{inv} t(z) = z' \qquad \text{whenever } z \in E_k,$$

and for i a positive integer, we define the functions F, F_i, $\bar{F}$, $\bar{F}_i$ by the formulae

$$F = f \circ t, \qquad F_i = f_i \circ t,$$
$$\bar{F}(x) = (x_1, x_2, \cdots, x_k, F(x)) \qquad \text{for } x \in E_k,$$
$$\bar{F}_i(x) = (x_1, x_2, \cdots, x_k, F_i(x)) \qquad \text{for } x \in E_k.$$

The rest of the proof is divided into three parts.

PART 1. *If* $V \in G_k$, $B \subset E_k$, *and* $z \in E_k$, *then*

$$[(\text{inv } V)^*(B)]_z = (\text{inv } V)^*[B_{V(z)}].$$

Proof.

$$\begin{aligned}(\text{inv } V)^*[B_{V(z)}] &= (\text{inv } V)^*(\{y \mid y - V(z) \in B\}) \\ &= \{x \mid V(x) - V(z) \in B\} \\ &= \{x \mid x - z \in (\text{inv } V)^*(B)\} \\ &= [(\text{inv } V)^*(B)]_z.\end{aligned}$$

PART 2. (i) $p^k_{k+1} \circ R \circ \bar{f} = U \circ p^k_{k+1} \circ S \circ \bar{F} \circ \text{inv } t$,
(ii) $p^k_{k+1} \circ R \circ \bar{f}_i = U \circ p^k_{k+1} \circ S \circ \bar{F}_i \circ \text{inv } t$.
Proof. It is sufficient to verify (ii). Let $z \in E_k$, then

$$\begin{aligned}(p^k_{k+1} \circ R \circ \bar{f}_i)(z) &= (p^k_{k+1} \circ R)(z_1, z_2, \cdots, z_k, f_i(z)) \\ &= (U \circ p^k_{k+1} \circ S \circ \text{inv } T)(z_1, z_2, \cdots, z_k, f_i(z)) \\ &= (U \circ p^k_{k+1} \circ S)(z_1', z_2', \cdots, z_k', f_i(z)) \\ &= (U \circ p^k_{k+1} \circ S)(z_1', z_2', \cdots, z_k', F_i(z')) \\ &= (U \circ p^k_{k+1} \circ S \circ \bar{F}_i)(z') \\ &= (U \circ p^k_{k+1} \circ S \circ \bar{F}_i \circ \text{inv } t)(z).\end{aligned}$$

PART 3.

$$\int_{E_k} N(p^k_{k+1} \circ R \circ \bar{f}_i, Y, x)d\mathcal{L}_k x$$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(p^k_{k+1} \circ S \circ \bar{f}, Y_z, x)d\mathcal{L}_k x d\mathcal{L}_k z.$$

Proof. With the aid successively of Part 2, Lemma 3.2 applied to $\bar{F}_i$, Part 1, the fact that

$$Jt(z) = 1 \qquad \text{whenever } z \in E_k,$$

and Part 2 again, we check that

$$\int_{E_k} N(p^k_{k+1} \circ R \circ \bar{f}_i, Y, x)d\mathcal{L}_k x$$

$$= \int_{E_k} N(U \circ p^k_{k+1} \circ S \circ \bar{F}_i \circ \text{inv } t, Y, x)d\mathcal{L}_k x$$

$$= \int_{E_k} N(p^k_{k+1} \circ S \circ \bar{F}_i \circ \text{inv } t, Y, x)d\mathcal{L}_k x$$

$$= \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F}_i, (\text{inv } t)^*(Y), x) d\mathcal{L}_k x$$

$$\leqq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F}, [(\text{inv } t)^*(Y)]_z, x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F}, (\text{inv } t)^*[Y_{t(z)}], x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y_{t(z)}, x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} Jt(z) \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y_{t(z)}, x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{t^*(K_i)} \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y_z, x) d\mathcal{L}_k x d\mathcal{L}_k z$$

$$= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\mathcal{L}_k z.$$

3.5 THEOREM. *If Y is a bounded open convex subset of E_k, then*

$$R \in G_{k+1} \text{ implies } N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) = S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x)$$

for $\mathcal{L}_k$ almost all $x \in E_k$.

Proof. Let $R \in G_{k+1}$. Just as in the preceding lemma we may select functions U, S, t and define functions F and $\overline{F}$ so that

$$S \in \Lambda_{k+1}, \qquad U \in G_k, \qquad t \in G_k,$$
$$F = f \circ t,$$
$$\overset{k}{p}_{k+1} \circ R \circ \bar{f} = U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t.$$

Then with the help of Lemma 3.2 we find that

$$\begin{aligned} &N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) \\ &\quad = N(U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y, x) \\ &\quad = N(\overset{k}{p}_{k+1} \circ S \circ \overline{F}, (\text{inv } t)^*(Y), x) \\ &\quad = S(\overset{k}{p}_{k+1} \circ S \circ \overline{F}, (\text{inv } t)^*(Y), x) && \text{for } \mathcal{L}_k \text{ almost all } x \text{ in } E_k \\ &\quad = S(U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \text{inv } t, Y, x) && \text{for } \mathcal{L}_k \text{ almost all } x \text{ in } E_k \\ &\quad = S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) && \text{for } \mathcal{L}_k \text{ almost all } x \text{ in } E_k. \end{aligned}$$

3.6 REMARK. *If $Y \subset E_k$ is the interior of a k cell and $R \in G_{k+1}$, then*

$$\int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x = \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x$$
$$= L(\overset{k}{p}_{k+1} \circ R \circ \bar{f} \mid Y).$$

For if S, U, t, and F are chosen as in Theorem 3.5, it then follows by Theorems 2.6 and 3.5 and certain invariance properties of the Lebesgue area that

$$L(\overset{k}{p}_{k+1} \circ R \circ \bar{f} \mid Y) = L(U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \operatorname{inv} t \mid Y) = L(\overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \operatorname{inv} t \mid Y)$$
$$= L(\overset{k}{p}_{k+1} \circ S \circ \overline{F} \mid (\operatorname{inv} t)^*(Y))$$
$$= \int_{E_k} N(\overset{k}{p}_{k+1} \circ S \circ \overline{F}, (\operatorname{inv} t)^*(Y), x) d\mathcal{L}_k x$$
$$= \int_{E_k} N(U \circ \overset{k}{p}_{k+1} \circ S \circ \overline{F} \circ \operatorname{inv} t, Y, x) d\mathcal{L}_k x$$
$$= \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x$$
$$= \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x.$$

3.7 LEMMA. *If $X \subset E_k$ is a k cell with boundary $\dot{X}$, then*

$$\overset{k}{\mathcal{H}}_{k+1}[\bar{f}^*(\dot{X})] = 0.$$

Proof. The method of proof selected for Lemma 2.5 included an explicit proof of this statement.

3.8 THEOREM. *If $X \subset E_k$ is a k cell, then*

$$\beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, X, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, X, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$= L(\bar{f} \mid X)$$
$$= \lim_{j \to \infty} L(\bar{f}_j \mid X).$$

Proof. Let Y=interior X. Then from Lemma 3.5 we know that

$$\int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x = \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x$$

for $R \in G_{k+1}$. Therefore

$$\beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.$$

We suppose that A is an open interval of E_k for which

$$\text{closure } A \subset Y,$$

and select a positive integer i so large that

$$A_z \subset Y \qquad \text{whenever } z \in K_i.$$

Then with the help of [F4, 6.13], [F4, 4.5], and Lemma 3.4 we obtain

$$L(\bar{f}_i \mid \text{closure } A) = \int_A J\bar{f}_i(x) d\mathcal{L}_k x$$
$$= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}_i, A, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$\leqq \beta(k+1, k)^{-1} \int_{G_{k+1}} \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, A_z, x) d\mathcal{L}_k x d\mathcal{L}_k z d\phi_{k+1} R$$
$$\leqq \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.$$

From the lower semi-continuity of L on $C_k(X)$ it follows that

$$L(\bar{f} \mid \text{closure } A) \leqq \liminf_{i \to \infty} L(\bar{f}_i \mid \text{closure } A)$$
$$\leqq \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.$$

As a consequence of the arbitrary nature of A

$$L(\bar{f} \mid Y) \leqq \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R$$
$$\leqq L(\bar{f} \mid Y).$$

Because of Lemma 3.7 this last relationship holds when Y is replaced by X.

Using the foregoing results the other part of the statement may be proved exactly as the corresponding statement was proved in Theorem 2.6.

3.9 REMARK. For $X \subset E_k$ a k cell, the results of the last theorem allow us, using the notation of [F6], to state

$$L(\bar{f} \mid X) = M^{**}(\bar{f} \mid X) = S^{**}(\bar{f} \mid X) = U^{**}(\bar{f} \mid X) = V^{**}(\bar{f} \mid X) = N^{**}(\bar{f} \mid X).$$

3.10 REMARK. *If Y is an analytic subset of E_k, then*

$$\mathcal{J}_{k+1}^{k}[\bar{f}^*(Y)] = \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.$$

Since $\bar{f}^*(Y)$ is an analytic subset of E_{k+1}, the univalence of $\bar{f}$ implies

$$\begin{aligned} \mathcal{J}_{k+1}^{k}[\bar{f}^*(Y)] &= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R, \bar{f}^*(Y), x) d\mathcal{L}_k x d\phi_{k+1} R \\ &= \beta(k+1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R. \end{aligned}$$

3.11 THEOREM. *If $X \subset E_k$ is a k cell and A is a Borel subset of E_{k+1}, then*

$$\mathcal{J}_{k+1}^{k}[A \cap \bar{f}^*(X)] = \Gamma_{k+1}^{k}[A \cap \bar{f}^*(X)].$$

Proof. Since in general $\mathcal{J}_{k+1}^{k}$ is dominated by Γ_{k+1}^{k}, it is sufficient to assume

$$\mathcal{J}_{k+1}^{k}[A \cap \bar{f}^*(X)] < \infty,$$

and to show that

$$\mathcal{J}_{k+1}^{k}[A \cap \bar{f}^*(X)] \geqq \Gamma_{k+1}^{k}[A \cap \bar{f}^*(X)].$$

We divide the proof of this into three parts.

PART 1. *If $R \in G_{k+1}$, then*

$$L(\bar{f} \mid X) \geqq \int_{E_k} S(\overset{k}{p}_{k+1} \circ R \circ \bar{f}, X, x) d\mathcal{L}_k x.$$

Proof. Let $R \in G_{k+1}$. Select a sequence

$$P_1, P_2, \cdots,$$

of polyhedra of $C_k(X)$ for which

$$\lim_{i \to \infty} P_i = \bar{f} \mid X, \qquad \liminf_{i \to \infty} L(P_i) = L(\bar{f} \mid X).$$

For each positive integer i, $\overset{k}{p}_{k+1} \circ R \circ P_i$ is a polyhedron and

$$\lim_{i \to \infty} (\overset{k}{p}_{k+1} \circ R \circ P_i) = \overset{k}{p}_{k+1} \circ R \circ \bar{f} \mid X.$$

Suppose $\bigcup_{j=1}^{m_i} T_j$ is the simplicial triangulation of X associated with P_i.

From the Lipschitzian character of $p_{k+1}^k \circ R$ we may infer that

$$\begin{aligned} L(\bar{f} \mid X) &= \liminf_{i\to\infty} \sum_{j=1}^{m_i} \mathcal{H}_{k+1}^k[\overset{*}{P_i}(T_j)] \\ &\geqq \liminf_{i\to\infty} \sum_{j=1}^{m_i} \mathcal{H}_{k+1}^k[(p_{k+1}^k \circ R \circ P_i)^*(T_j)] \\ &\geqq L(p_{k+1}^k \circ R \circ \bar{f} \mid X). \end{aligned}$$

We know that the k-dimensional Lebesgue area and the k-dimensional stable area are lower semi-continuous functions on $C_k(X)$, and that both are extensions of the classical area integral over the class of polyhedra. The Lebesgue area being numerically the largest of all such lower semi-continuous extensions, we may conclude that

$$L(p_{k+1}^k \circ R \circ \bar{f} \mid X) \geqq \int_{E_k} S(p_{k+1}^k \circ R \circ \bar{f},\, X,\, x)\, d\mathcal{L}_k x.$$

PART 2. *If B is a Borel subset of E_k and $R \in G_{k+1}$, then*

$$\mathcal{J}_{k+1}^k[\bar{f}^*(B)] \geqq \int_{E_k} N(p_{k+1}^k \circ R \circ \bar{f},\, B,\, x)\, d\mathcal{L}_k x.$$

Proof. Using Remark 3.10, Theorem 3.8, and Part 1,

$$\begin{aligned} \mathcal{J}_{k+1}^k[\bar{f}^*(Y)] = L(\bar{f} \mid Y) &\geqq \int_{E_k} S(p_{k+1}^k \circ R \circ \bar{f},\, Y,\, x)\, d\mathcal{L}_k x \\ &= \int_{E_k} N(p_{k+1}^k \circ R \circ \bar{f},\, Y,\, x)\, d\mathcal{L}_k x \end{aligned}$$

whenever $Y \subset E_k$ is a k cell.

The inequality between the first and last members of this string may be extended (by the customary methods) to hold whenever Y is a Borel subset of E_k.

PART 3. $\mathcal{J}_{k+1}^k[A \cap \bar{f}^*(X)] \geqq \Gamma_{k+1}^k[A \cap \bar{f}^*(X)]$.

Proof. Suppose B is a Borel subset of E_{k+1} and $B' = (\text{inv } \bar{f})^*[B \cap \bar{f}^*(X)]$. Then using Part 2

$$\begin{aligned} \mathcal{J}_{k+1}^k[B \cap \bar{f}^*(X)] = \mathcal{J}_{k+1}^k[\bar{f}^*(B')] &\geqq \int_{E_k} N(p_{k+1}^k \circ R \circ \bar{f},\, B',\, x)\, d\mathcal{L}_k x \\ &\geqq \mathcal{L}_k[(p_{k+1}^k \circ R \circ \bar{f})^*(B')] \\ &= \mathcal{L}_k[(p_{k+1}^k \circ R)^*(B \cap \bar{f}^*(X))] \qquad \text{for } R \in G_{k+1}, \end{aligned}$$

$$\mathcal{J}_{k+1}^k[B \cap \bar{f}^*(X)] \geqq \gamma_{k+1}^k[B \cap \bar{f}^*(X)].$$

It follows that

$$\mathcal{J}_{k+1}^{k}[A \cap \bar{f}^*(X)] \geqq \Gamma_{k+1}^{k}[A \cap \bar{f}^*(X)].$$

3.12 LEMMA. *If B is a Borel subset of E_k and $R \in G_{k+1}$, then*

$$\int_{E_k} N(p_{k+1}^{k} \circ R \circ \bar{f}, B, y)\, d\mathcal{L}_k y \leqq \mathcal{J}_{k+1}^{k}[\bar{f}^*(B)]$$

$$\leqq \sum_{i=1}^{k+1} \int_{E_k} N(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f}, B, y)\, d\mathcal{L}_k y.$$

Proof. The first inequality is a restatement of Part 2 of the preceding theorem.

Let $Y \subset E_k$ be a k cell with boundary $\dot{Y}$. With the help of [F4, 6.13] we infer that

$$\begin{aligned} L(\bar{f}_j \mid Y) &= \int_{Y-\dot{Y}} J\bar{f}_j(x)\, d\mathcal{L}_k x \\ &= \int_{Y-\dot{Y}} \left[\sum_{i=1}^{k+1} \{J(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f}_j)(x)\}^2 \right]^{1/2} d\mathcal{L}_k x \\ &\leqq \int_{Y-\dot{Y}} \sum_{i=1}^{k+1} J(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f}_j)(x)\, d\mathcal{L}_k x \\ &= \sum_{i=1}^{k+1} L(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f}_j \mid Y) \end{aligned}$$

whenever j is a positive integer. Letting $j \to \infty$, it follows from Theorems 3.10 and 2.8 that $L(\bar{f} \mid Y) \leqq \sum_{i=1}^{k+1} L(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f} \mid Y)$. We see by reference to Theorems 2.6, 3.8, and Remark 3.10 that

$$\mathcal{J}_{k+1}^{k}[\bar{f}^*(Y)] \leqq \sum_{i=1}^{k+1} \int_{Y} N(p_{k+1}^{k} \circ {}^{i}R \circ \bar{f}, Y, y)\, d\mathcal{L}_k y.$$

The proof is completed by extending this inequality to hold whenever Y is a Borel subset of E_k.

3.13 SECTIONAL ASSUMPTION. For the rest of this section $X \subset E_k$ will denote a k cell.

3.14 LEMMA. *If j is a positive integer not exceeding k, then*

$$\int_{X(j)} T_{v=\inf X(j)^u}^{\sup X(j)^u} f(u_1, \cdots, u_{j-1}, v, u_{j+1}, \cdots, u_k)\, d\mathcal{L}_{k-1} u$$

$$\leqq L(\bar{f} \mid X) \leqq \sum_{i=1}^{k} \int_{X(i)} T_{v=\inf X(i)^u}^{\sup X(i)^u} f(u_1, \cdots, u_{i-1}, v, u_{i+1}, \cdots, u_k)\, d\mathcal{L}_{k-1} u$$

$$+ \mathcal{L}_k(X).$$

Proof. We recall for $i=1, 2, \cdots, k$, that

$$(\overset{k}{p}_{k+1} \circ {}^{i}R \circ \bar{f})(x) = (x_1, \cdots, x_{i-1}, f(x), x_{i+1}, \cdots, x_k) \qquad \text{for } x \in E_k,$$
$$\overset{k}{p}_{k+1} \circ {}^{i}R \circ \bar{f} \in \Omega_k.$$

Consequently we may use Part 2 of Lemma 2.2, Theorem 3.8, and the preceding lemma to compute:

$$\int_{X(j)} T^{\sup X(j)^u}_{v=\inf X(j)^u} f(u_1, \cdots, u_{j-1}, v, u_{j+1}, \cdots, u_k)d\mathcal{L}_{k-1}u$$
$$= \int_{E_k} N(\overset{k}{p}_{k+1} \circ {}^{j}R \circ \bar{f}, X, x)d\mathcal{L}_k x \leqq \overset{k}{\mathcal{J}}_{k+1}[\bar{f}^*(X)]$$
$$= L(\bar{f} \mid X) \leqq \sum_{i=1}^{k+1} \int_{E_k} N(\overset{k}{p}_{k+1} \circ {}^{i}R \circ \bar{f}, X, x)d\mathcal{L}_k x$$
$$= \sum_{i=1}^{k} \int_{X(i)} T^{\sup X(i)^u}_{v=\inf X(i)^u} f(u_1, \cdots, u_{i-1}, v, u_{i+1}, \cdots, u_k)d\mathcal{L}_{k-1}u + \mathcal{L}_k(X).$$

3.15 THEOREM. *The following statements are equivalent:*
(i) *f is BVT on X.*
(ii) $L(\bar{f}|X) < \infty$.

Proof. This statement is an immediate consequence of the preceding lemma.

3.16 THEOREM. *If f is BVT on X and U is the subset of X on which $\bar{f}$ is approximately differentiable, then*
(i) $\mathcal{L}_k(X-U)=0$,
(ii) $\int_Y J\bar{f}(x)d\mathcal{L}_k x = \mathcal{J}^k_{k+1}[\bar{f}^*(Y \cap U)] \leqq \mathcal{J}^k_{k+1}[\bar{f}^*(Y)] < \infty$
whenever Y is an $\mathcal{L}_k$ measurable subset of X.

Proof. Let j be a positive integer no greater than k. Just as in Theorem 2.14 we can show that

$$D_j f(x) \text{ exists for } \mathcal{L}_k \text{ almost all } x \text{ in } X.$$

Whence

$$D_j \bar{f}(x) \text{ exists for } \mathcal{L}_k \text{ almost all } x \text{ in } X,$$

and Stepanoff's theorem (see [S, IX 12.2]) implies that $\mathcal{L}_k(X-U)=0$. We may apply [F2, 5.2], with the measure Φ replaced by $\mathcal{J}^k_{k+1}$ (see [F5, 5.10]), and Theorem 3.15 to obtain

$$\int_Y J\bar{f}(x)d\mathcal{L}_k x = \int_{Y \cap U} J\bar{f}(x)d\mathcal{L}_k x = \int_{E_k} N(\bar{f}, Y \cap U, y)d\overset{k}{\mathcal{J}}_{k+1}y$$
$$= \overset{k}{\mathcal{J}}_{k+1}[\bar{f}^*(Y \cap U)] \leqq \overset{k}{\mathcal{J}}_{k+1}[\bar{f}^*(Y)] < \infty.$$

3.17 THEOREM. *If f is BVT on X, then the following statements are equivalent*:

(i) $\mathcal{J}_{k+1}^k[\bar{f}^*(V)]=0$ *whenever* $V\subset X$ *and* $\mathcal{L}_k(V)=0$,

(ii) $\int_Y J\bar{f}(x)d\mathcal{L}_k x=\mathcal{J}_{k+1}^k[\bar{f}^*(Y)]<\infty$ *whenever Y is an $\mathcal{L}_k$ measurable subset of X*,

(iii) $\int_X J\bar{f}(x)d\mathcal{L}_k x=\mathcal{J}_{k+1}^k[\bar{f}^*(X)]<\infty$,

(iv) *f is ACT on X.*

Proof. Let $V\subset X$ and $\mathcal{L}_k(V)=0$. Select a Borel subset A of E_k so that

$$V \subset A \subset X, \qquad \mathcal{L}_k(A) = 0.$$

The proof will be divided into five parts.

PART 1. (i) *implies* (ii).

Proof. Let Y be an $\mathcal{L}_k$ measurable subset of X and U be the subset of X on which $\bar{f}$ is approximately differentiable. Then

$$\mathcal{L}_k(Y - U) = 0,$$
$$\mathcal{J}_{k+1}^k[\bar{f}^*(Y - U)] = 0,$$

and Theorem 3. 16 implies

$$\int_Y J\bar{f}(x)d\mathcal{L}_k x = \mathcal{J}_{k+1}^k[\bar{f}^*(Y \cap U)] + \mathcal{J}_{k+1}^k[\bar{f}^*(Y - U)]$$
$$= \mathcal{J}_{k+1}^k[\bar{f}^*(Y)] < \infty.$$

PART 2. (ii) *implies* (iii).

PART 3. (iii) *implies* (i).

Proof. Let $\epsilon>0$. Choose on X a non-negative continuous function c and a number M satisfying

$$\int_X | Jf(x) - c(x) | d\mathcal{L}_k x \leqq \epsilon, \qquad M > \sup_{x\in X} c(x).$$

We can select an open set G of E_k for which

$$V \subset G, \qquad \mathcal{L}_k(G) < \epsilon\cdot M^{-1},$$

and a grating of $k-1$ planes which defines such a family $\{X_i | i=1, 2, \cdots\}$ of subsets of E_k that

$$V \subset \bigcup_{i=1}^{\infty} X_i \subset G \cap X, \qquad X_i \text{ is a } k \text{ cell for } i = 1, 2, \cdots,$$
$$\text{interior } X_i \cap \text{interior } X_j = 0 \qquad \text{for } i \neq j.$$

Then

$$\mathcal{J}_{k+1}^{k}[\bar{f}^*(V)] \leqq \mathcal{J}_{k+1}^{k}\left[\bar{f}^*\left(\bigcup_{i=1}^{\infty} X_i\right)\right] \leqq \sum_{i=1}^{\infty} \mathcal{J}_{k+1}^{k}[\bar{f}^*(X_i)]$$

$$= \sum_{i=1}^{\infty} \int_{X_i} J\bar{f}(x) d\mathcal{L}_k x = \sum_{i=1}^{\infty} \int_{\text{interior } X_i} J\bar{f}(x) d\mathcal{L}_k x$$

$$\leqq \int_{G \cap X} J\bar{f}(x) d\mathcal{L}_k x \leqq \int_{G \cap X} c(x) d\mathcal{L}_k x + \epsilon < 2\epsilon.$$

Because of the arbitrary nature of ϵ the proof is complete.

PART 4. (ii) *implies* (iv).

Proof. Suppose j is a positive integer no greater than k.

The truth of (ii) means

$$\mathcal{J}_{k+1}^{k}[\bar{f}^*(A)] = 0,$$

and it follows from Lemma 3.12 and Theorem 2.15 that

$$\mathcal{L}_k[(\overset{k}{p}_{k+1} \circ {}^{j}R \circ \bar{f})^*(A)] = 0, \qquad \mathcal{L}_k[(\overset{k}{p}_{k+1} \circ {}^{j}R \circ \bar{f})^*(V)] = 0,$$
$$\overset{k}{p}_{k+1} \circ {}^{j}R \circ \bar{f} \text{ is absolutely continuous on } X,$$
$$f \text{ is ACT } (j) \text{ on } X.$$

Consequently

$$f \text{ is ACT on } X.$$

PART 5. (iv) *implies* (i).

Proof. Using Theorem 2.15 we know that

$$f \text{ is ACT } (i) \text{ on } X,$$
$$\overset{k}{p}_{k+1} \circ {}^{i}R \circ \bar{f} \text{ is absolutely continuous on } X,$$
$$\int_{E_k} N(\overset{k}{p}_{k+1} \circ {}^{i}R \circ \bar{f}, A, x) d\mathcal{L}_k x = 0,$$

for $i = 1, 2, \cdots, k$. Recalling that

$$\int_{E_k} N(\overset{k}{p}_{k+1} \circ {}^{k+1}R \circ \bar{f}, A, x) d\mathcal{L}_k x = \mathcal{L}_k(A),$$

we may conclude from Lemma 3.12 that

$$\mathcal{J}_{k+1}^{k}[\bar{f}^*(A)] = 0, \qquad \mathcal{J}_{k+1}^{k}[\bar{f}^*(V)] = 0.$$

3.18 REMARK. *If*

(i) *h is a continuous function on X into X,*

(ii) $\int_{E_k} N(h, X, x) d\mathcal{L}_k x < \infty$,

(iii) *f is ACT on X,*

then h is absolutely continuous on X if and only if

$$\mathcal{J}^k_{k+1}[(\bar{f}\circ h)^*(V)] = 0 \textit{ whenever } V \subset X \textit{ and } \mathcal{L}_k(V) = 0.$$

This is a consequence of Theorem 3.17.

3.19 REMARK. Using the notation of [F6] we may restate the results of Theorems 3.8, 3.16, and 3.17:

$$L(\bar{f}\mid X) < \infty \textit{ implies}$$

$$\int_X J\bar{f}(x)d\mathcal{L}_k x \leqq L(\bar{f}\mid X) = M^{**}(\bar{f}\mid X) = S^{**}(\bar{f}\mid X) = U^{**}(\bar{f}\mid X)$$
$$= V^{**}(\bar{f}\mid X) = N^{**}(\bar{f}\mid X),$$

equality holding if and only if f is ACT on X.

3.20 THEOREM. *If for $x\in E_k$ and $r>0$*

$$C(x, r) = E_k \cap \{z \mid |z - x| \leqq r\},$$

then

$$f \textit{ is BVT on } X \textit{ implies}$$

$$\lim_{r\to 0+} \frac{L[\bar{f}\mid C(x, r)]}{\alpha(k)r^k} = J\bar{f}(x) \qquad \textit{for } \mathcal{L}_k \textit{ almost all } x \in \textit{interior } X.$$

Proof. Letting Theorems 3.8 and 3.16 play the respective roles of Theorems 2.6 and 2.14, the proof is similar to that of Theorem 2.18. In fact only the verification of the additivity of the singular part of the decomposition is different. For this the following general property of Carathéodory outer measures proves useful:

if ϕ is a Carathéodory outer measure, A, B, and T are elements of domain ϕ, and A is ϕ measurable, then

$$\phi(T\cap(A\cup B)) + \phi(T\cap A\cap B) = \phi(T\cap A) + \phi(T\cap B).$$

BIBLIOGRAPHY

C. CARATHÉODORY

[C] *Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs,* Nachr. Ges. Wiss. Göttingen (1914) pp. 404–426.

H. FEDERER

[F1] *Surface area.* I, Trans. Amer. Math. Soc vol. 55 (1944) pp. 420–437.

[F2] *Surface area.* II, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 438–456.

[F3] *The Gauss-Green theorem,* Trans. Amer. Math. Soc. vol. 58 (1945) pp. 44–76.

[F4] *Coincidence functions and their integrals,* Trans. Amer. Math. Soc. vol. 59 (1946) pp. 441–466.

[F5] *The (ϕ, k) rectifiable subsets of n space,* Trans. Amer. Math. Soc. vol. 62 (1947) pp. 114–192.

[F6] *Hausdorff measure and Lebesgue area*, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 90–94.

[F7] *Measure and area*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 306–378.

F. Hausdorff

[H] *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1918) pp. 157–179.

P. V. Reichelderfer

[R] *On bounded variation and absolute continuity for parametric representation of continuous surfaces*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 251–291.

S. Saks

[S] *Theory of the integral*, Warsaw, 1937.

A. Sard

[SD] *The equivalence of n-measure and Lebesgue measure in E_n*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 758–759.

L. Tonelli

[T1] *Sulla quadratura della superficie*, Rendiconti Acc. Lincei (6) vol. 3_1 (1926) pp. 357–362, pp. 445–450, pp. 633–638, pp. 714–719.

[T2] *Su un polinomio d'approssimazione e l'area di una superficie*, Rendiconti Acc. Lincei (6) vol. 5 (1927) pp. 313–318.

A. Weil

[W] *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, vol. 869, Paris, Hermann, 1938.

Brown University,
Providence, R. I.